Common fixed point theorems in digital metric spaces

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Abstract

In this paper, we introduce commutative, weakly commutative, compatible, weakly compatible maps in the setting of digital metric space and prove some common fixed point results for these mappings. We also give some examples to illustrate our results. Also, an application of the results is proposed for the digital images.

Keywords: Digital metric space; commutative maps; compatible maps; weakly commutative maps; weakly compatible maps; common fixed point theorems.

1. Introduction

The dawn of the fixed point theory starts when in 1912 Brouwer proved a fixed point result for continuous self maps on a closed ball. In 1922, Banach [2] gave a very useful result known as the Banach Contraction Principle. After which a lot of implications of Banach contraction came into existence ([1,8,9,20,21]).

A major shift in the arena of fixed point theory came in 1976 when Jungck [16], defined the concept of commutative maps and proved the common fixed point results for such maps. After which, Sessa[27] gave the concept of weakly compatible, and Jungck ([17,18])gave the concepts of compatibility and weak compatibility. Certain altercations of commutativity and compatibility can also be found in [10,19,24,25,28].

Digital topology is a developing area based on general topology and functional analysis which studies features of 2D and 3D digital images. Rosenfield[26] was the first to consider digital topology as the tool to study digital images. Kong[23], then introduced the digital fundamental group of a discrete object. The digital versions of the topological concepts were given by Boxer[3], who later studied digital continuous functions[4]. Later, he gave results of digital homology groups of 2D digital images in [6] and [7]. Ege and Karaca [12,13] give relative and reduced Lefschetz fixed point theorem for digital images. They also calculate degree of antipodal map for the sphere like digital images using fixed point properties. Ege and Karaca [14] then defined a digital metric space and proved the famous Banach Contraction Principle for digital images.

2. Preliminaries

Let X be a subset of $\mathbb{Z}^n$ for a positive integer n where $\mathbb{Z}^n$ is the set of lattice points in the n-dimensional Euclidean space and ρ represent an adjacency relation for the members of X. A digital image consists of (X, ρ).

Definition 2.1[5]: Let l, n be positive integers, $1 \leq l \leq n$ and two distinct points

$a = (a_1, a_2, \ldots, a_n), b = (b_1, b_2, \ldots, b_n) \in \mathbb{Z}^n$

a and b are k_l- adjacent if there are at most l indices i such that $|a_i - b_i| = 1$ and for all other indices j such that $|a_j - b_j| \neq 1$, $a_j = b_j$.

A ρ-neighbour [5] of $a \in \mathbb{Z}^n$ is a point of $\mathbb{Z}^n$ that is ρ- adjacent to a where $\rho \in \{2,4,8,6,18,26\}$ and $n \in 1,2,3$. The set

$$\mathrm{N}_\rho(a) = \{b | b \text{ } is \text{ } \rho - adjacent \text{ } to \text{ } a\}$$

is called the ρ- neighbourhood of a. A digital interval [12] is defined by

$$[p, q]_{\mathbb{Z}} = \{z \in \mathbb{Z} | p \leq z \leq q\}$$

where $p, q \in \mathbb{Z}$ and $p < q$.

A digital image $X \subset \mathbb{Z}^n$ is ρ- connected [15] if and only if for every pair of different points $u, v \in X$, there is a set $\{u_0, u_1, \ldots, u_r\}$ of points of digital image X such that $u = u_0, v = u_r$ and u_i and u_{i+1} are ρ- neighbours where $i = 0,1, \ldots, r-1$.

Definition 2.2: Let $(X, \rho_0) \subset \mathbb{Z}^{n_0}$, $(Y, \rho_1) \subset \mathbb{Z}^{n_1}$ be digital images and $T: X \to Y$ be a function.

- T is said to be (ρ_0, ρ_1)- continuous[5], if for all ρ_0- connected subset E of X, $f(E)$ is a ρ_1- connected subset of Y.
- For all ρ_0- adjacent points $\{u_0, u_1\}$ of X, either $T(u_0) = T(u_1)$ or $T(u_0)$ and $T(u_1)$ are a ρ_1- adjacent in Y if and only if T is (ρ_0, ρ_1)- continuous [5].
- If f is (ρ_0, ρ_1)- continuous, bijective and T^{-1} is (ρ_1, ρ_0)- continuous, then T is called (ρ_0, ρ_1)- isomorphism [6]and denoted by $X \cong_{(\rho_0, \rho_1)} Y$.

A $(2, \rho)$- continuous function T, is called a digital ρ- path [5] from u to v in a digital image X if $T: [0, m]_{\mathbb{Z}} \to X$ such that $T(0) = u$ and $T(m) = v$. A simple closed ρ- curve of $m \geq 4$ points [7] in a digital image X is a sequence $\{T(0), T(1), \ldots, T(m-1)\}$ of images of the ρ-path $T: [0, m-1]_{\mathbb{Z}} \to X$ such that $T(i)$ and $T(j)$ are ρ- adjacent if and only if $j = i \pm mod \text{ } m$.

Definition 2.5 [14]: A sequence $\{x_n\}$ of points of a digital metric space (X, d, ρ) is a Cauchy sequence if for all $\in > 0$, there exists $\delta \in \mathbb{N}$ such that for all $n, m > \delta$, then

$$d(x_n, x_m) < \in.$$

Definition 2.6 [14]: A sequence $\{x_n\}$ of points of a digital metric space (X, d, ρ) converges to a limit $p \in X$ if for all $\in > 0$, there exists $\alpha \in \mathbb{N}$ such that for all $n > \delta$, then

$$d(x_n, p) < \in.$$

Definition 2.7 [14]: A digital metric space (X, d, ρ) is a digital metric space if any Cauchy sequence $\{x_n\}$ of points of (X, d, ρ) converges to a point p of (X, d, ρ).

Definition 2.8 [14]: Let (X, ρ) be any digital image. A function $T: (X, \rho) \to (X, \rho)$ is called right- continuous if $f(p) = \lim_{x \to p^+} T(x)$ where, $p \in X$.

Definition 2.9 [14]: Let, (X, d, ρ) be any digital metric space and $T: (X, d, \rho) \to (X, d, \rho)$ be a self digital map. If there exists $\alpha \in (0,1)$ such that for all $x, y \in X$,

$$d\big(f(x), f(y)\big) \leq \alpha d(x, y),$$

then T is called a digital contraction map.

Proposition 2.10 [14]: Every digital contraction map is digitally continuous.

Theorem 2.11 [14]: (Banach Contraction principle) Let (X, d, ρ) be a complete metric space which has a usual Euclidean metric in $\mathbb{Z}^n$. Let, $T: X \to X$ be a digital contraction map. Then T has a unique fixed point, i.e. there exists a unique $p \in X$ such that $f(p) = p$.

3. Main Results

3.1. Commutative maps:

Definition 3.1.1: Suppose that (X, d, ρ) is a complete digital metric space and $S, T : X \to X$ be maps defined on X. Then $S \; and \; T$ are said to be commutative if $S \circ T(x) = T \circ S(x) \; \forall \, x \in X$.

Proposition 3.1.2: Let T be a mapping of X into itself. Then T has a fixed point iff there is a constant map $S: X \to X$ which commutes with T.

Proof: By hypothesis there exists $a \in X$ and $S: X \to X$ such that $S(x) = a$ and $S\big(T(x)\big) = T\big(S(x)\big)$ for all $x \in X$. We can therefore write $T(a) = T\big(S(a)\big) = S\big(T(a)\big) = a$, so that a fixed point of T.

The condition of necessity is proved along with theorem 3.1.4.

Lemma 3.1.3: Let $\{y_n\}$ be a sequence of complete digital metric space (X, d, ρ). If there exists $\alpha \in (0,1)$ such that $d(y_{n+1}, y_n) \leq \alpha d(y_n, y_{n-1})$ for all n, then $\{y_n\}$ converges to a point in X.

Proof: Direct consequence of Theorem 2.11.

Theorem 3.1.4: Let T be a continuous mapping of a complete digital metric space (X, d, ρ) into itself. Then T has a fixed point in X if and only if there exists an $\alpha \in (0,1)$ and a mapping $S: X \to X$ which commutes with T and satisfies

$$S(X) \subset T(X) \; and \; d\big(S(x), S(y)\big) \leq \alpha d\big(T(x), T(y)\big) \; for \; all \; x, y \in X \tag{1}$$

Indeed T and S have a unique common fixed point if (1) holds.

Proof: To see that the condition stated is necessary, suppose that $T(a) = a$ for some $a \in X$. Define $S: X \to X$ by $S(x) = a$ for all $x \in X$. Then $S\big(T(x)\big) = a$ and $T\big(S(x)\big) = T(a) =$

a $(\forall x \in X)$, so $S(T(x)) = T(S(x))$ for all $x \in X$ and T commutes with S. Moreover, $S(x) = a = T(a)$ for all $x \in X$ so that $S(x) \subset T(X)$. Finally, for any $\alpha \in (0,1)$ we have for all $x, y \in X$,

$$d(S(x), S(y)) = d(a, a) = 0 \leq \alpha d(T(x), T(y)).$$

Thus, (1) holds.

On the other hand, suppose there is a map S of X into itself which commutes with T and for which (1) holds. We show that the condition is sufficient to ensure that T and S have a unique fixed point.

To this end, let $x_0 \in X$ and let x_1 be such that $T(x_1) = S(x_0)$. In general, choose x_n so that

$$T(x_n) = S(x_{n-1}) \tag{2}$$

We can do this since $S(X) \subset T(X)$. The relation (1) and (2) imply that $d_N(T(x_{n+1}), T(x_n)) \leq \alpha d_N(T(x_n), T(x_{n-1}))$ for all n. The lemma 3.1.3 yields $t \in X$ such that

$$T(x_n) \to t \tag{3}$$

But then (2) implies that

$$S(x_n) \to t \tag{4}$$

Now since T is continuous, (1) implies that both S and T are continuous. Hence, (3) and (4) demand that $S(T(x_n)) \to S(t)$. But S and T commute so that $S(T(x_n)) = T(S(x_n))$ for all n. Thus, $S(t) = T(t)$, and consequently $T(T(t)) = T(S(t)) = S(S(t))$ by commutativity. We can therefore infer

$$d\left(S(t), S(S(t))\right) \leq \alpha d\left(T(t), T(S(t))\right) = \alpha d\left(S(t), S(S(t))\right).$$

Hence, $d\left(S(t), S(S(t))\right)(1-\alpha) \leq 0$. Since $\alpha \in (0,1)$, $S(t) = S(S(t)) = T(S(t))$; i.e. $S(t)$ is common fixed point of T and S.

To see that T and S can have only one common fixed point, suppose that $x = T(x) = S(x)$ and $y = T(y) = S(y)$. Then (1) implies that $d(x, y) = d(S(x), S(y)) \leq \alpha d(T(x), T(y)) = \alpha d(x, y)$, or $d(x, y)(1-\alpha) \leq 0$. Since $\alpha < 1, x = y$.

Corollary 3.1.5: Let T and S be commuting mappings of a complete digital metric space (X, d, ρ) into itself. Suppose that T is continuous and $S(X) \subset T(X)$. If there exists $\alpha \in (0,1)$ and a positive integer k such that (i) $d_N\left(S^k(x), S^k(y)\right) \leq \alpha d_N(T(x), T(y))$ for all x and y in X, then T and S have a common fixed point.

Proof: Clearly, S^k commutes with T and $S^k(X) \subset S(X) \subset T(X)$. Thus the theorem pertains to S^k and T, so there exists a unique $a \in X$ such that $a = T(a) = S^k(a)$. But then, since T and S commute, we can write $S(a) = T\big(S(a)\big) = S^k\big(S(a)\big)$, which says that $S(a)$ is a c.f.p. of T and S^k. The uniqueness of a implies $a = S(a) = T(a)$.

Corollary 3.1.6: Let n be a positive integer and let K be a real number > 1. If S is a continuous mapping of a complete digital metric space (X, d, ρ) onto itself such that $d\big(S^n(x), S^n(y)\big) \geq Kd(x, y)$ for $x, y \in X$, then S has a unique fixed point.

Example 3.1.7: Let us consider the digital metric space (X, d, ρ), with the digital metric defined by $d(x, y) = |x - y|$. Consider the self mapping $T, S: X \to X$ given by, $T(x) = \frac{x}{2} + 3$ and $S(x) = \frac{x}{3} + 4$. Clearly, $S\big(T(x)\big) = \frac{x}{6} + 5 = T\big(S(x)\big)$. Also, $d\big(S(x), S(y)\big) \leq \frac{2}{3} d\big(T(x), T(y)\big)$. So, these maps satisfy all the hypothesis of theorem 3.1.4. So, S and T has a common fixed point. The common fixed point of S and T is $x = 6$.

3.2. Weakly commutative maps:

Definition 3.2.1: The self maps S and T of a non Newtonian metric space (X, d, ρ) are said to be weakly commutative iff $d\Big(S\big(T(x)\big), T\big(S(x)\big)\Big) \leq d\big(S(x), T(x)\big)$ for all $x \in X$.

Remark 3.2.2: Every pair of commutative maps is weakly commutative but the converse is not true. Moreover, the weakly commutative maps commute on the coincidences points.

If $t \in X$, be the coincidence point of commutative maps S and T, i.e., $S(t) = T(t)$.

Then, $d\Big(S\big(T(t)\big), T\big(S(t)\big)\Big) \leq d\big(S(t), T(t)\big) \quad \Rightarrow \quad d\Big(S\big(T(t)\big), T\big(S(t)\big)\Big) \leq d\big(S(t), S(t)\big)$ $\Rightarrow S\big(T(t)\big) = T\big(S(t)\big)$.

Theorem 3.2.3: Let T be a mapping of a complete digital metric space (X, d, ρ) into itself. Then T has a fixed point in X if and only if there exists an $\alpha \in (0,1)$ and a mapping $S: X \to X$ which commutes weakly with T and satisfies

$$S(X) \subset T(X) \text{ and } d\big(S(x), S(y)\big) \leq \alpha d\big(T(x), T(y)\big) \text{ for all } x, y \in X \tag{5}$$

Indeed T and S have a unique common fixed point if (5) holds.

Proof: To see that the condition stated is necessary, suppose that $T(a) = a$ for some $a \in X$. Define $S: X \to X$ by $S(x) = a$ for all $x \in X$. Then $S\big(T(x)\big) = a$ and $T\big(S(x)\big) = T(a) = a\ (\forall x \in X)$, so $S\big(T(x)\big) = T\big(S(x)\big)$ for all $x \in X$ and T commutes with S(Since commutativity implies weak commutativity.). So, S and T are weakly commutative. Moreover, $S(x) = a = T(a)$ for all $x \in X$ so that $S(x) \subset T(X)$. Finally, for any $\alpha \in (0,1)$ we have for all $x, y \in X$,

$$d\big(S(x), S(y)\big) = d(a, a) = 0 \leq \alpha d\big(T(x), T(y)\big).$$

Thus, (5) holds.

On the other hand, suppose there is a map S of X into itself which commutes weakly with T and for which (5) holds. We show that the condition is sufficient to ensure that T and S have a unique fixed point.

To this end, let $x_0 \in X$ and let x_1 be such that $T(x_1) = S(x_0)$. In general, choose x_n so that

$$T(x_n) = S(x_{n-1}) \tag{6}$$

We can do this since $S(X) \subset T(X)$. The relation (5) and (6) imply that $d\big(T(x_{n+1}), T(x_n)\big) \leq \alpha d\big(T(x_n), T(x_{n-1})\big)$ for all n. The lemma yields $t \in X$ such that

$$T(x_n) \to t \tag{7}$$

But then (6) implies that

$$S(x_n) \to t \tag{8}$$

Now, from the definition of weakly commutative

$$d\left(T\big(S(x_n)\big), S\big(T(x_n)\big)\right) \leq d\big(T(x_n), S(x_n)\big)$$

$$\Rightarrow d\big(T(t), S(t)\big) \leq d(t,t) \Rightarrow T(t) = S(t)$$

So, t is a coincidence point of S and T. So, $T\big(S(t)\big) = S\big(T(t)\big) = S\big(S(t)\big)$. We can therefore infer

$$d\left(S(t), S\big(S(t)\big)\right) \leq \alpha d\left(T(t), T\big(S(t)\big)\right) = \alpha d\left(S(t), S\big(S(t)\big)\right).$$

Hence, $d\left(S(t), S\big(S(t)\big)\right)(1-\alpha) \leq 0$. Since $\alpha \in (0,1)$, $S(t) = S\big(S(t)\big) = T\big(S(t)\big)$, i.e. $S(t)$ is common fixed point of S and T.

To see that S and T can have only one common fixed point, suppose that $x = T(x) = S(x)$ and $y = T(y) = S(y)$. Then (5) implies that $d(x,y) = d\big(S(x), S(y)\big) \leq \alpha d\big(T(x), T(y)\big) = \alpha d(x,y)$, or $d(x,y)(1-\alpha) \leq 0$. Since $\alpha < 1, x = y$.

Corollary 3.2.4: Let T and S be weakly commuting mappings of a complete digital metric space (X, d, ρ) into itself. Suppose, $S(X) \subset T(X)$. If there exists $\alpha \in (0,1)$ and a positive integer k such that (i) $d\big(S^k(x), S^k(y)\big) \leq \alpha d\big(T(x), T(y)\big)$ for all x and y in X, then T and S have a unique common fixed point.

Proof: Clearly, S^k commutes weakly with T and $S^k(X) \subset S(X) \subset T(X)$. Thus the theorem pertains to S^k and T, so there exists a unique $a \in X$ such that $a = T(a) = S^k(a)$. But then, since T and S commute weakly, we can write $S(a) = T\big(S(a)\big) = S^k\big(S(a)\big)$, which says that $S(a)$ is a c.f.p. of T and S^k. The uniqueness of a implies $a = S(a) = T(a)$.

Corollary 3.2.5: Let n be a positive integer and let K be a real number > 1. If S is a self map of a complete digital metric space (X, d, ρ) such that $d\big(S^n(x), S^n(y)\big) \geq Kd(x, y)$ for $x, y \in X$, then S has a unique fixed point.

3.3. Compatible Maps

Definition 3.3.1: The self maps S and T of a digital metric space are said to be compatible iff $\lim_{n\to\infty} d_N\left(S\big(T(x_n)\big), T\big(S(x_n)\big)\right) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T(x_n) = t$ for some t in X.

Remark 3.3.2: Now, maps which are commutative are clearly compatible but the converse is not true.

Lemma 3.3.3: Let $S, T: (X, d, \rho) \to (X, d, \rho)$ be continuous, and let $F = \{x \in X | S(x) = T(x) = x\}$. Then S and T are compatible if any one of the following conditions is satisfied:

(a) if $S(x_n), T(x_n) \to t (\in X)$, then $t \in F$.
(b) $d\big(S(x_n), T(x_n)\big) \to 0$ implies $D(S(x_n), F) \to 0$.

where $D(x, F) = \max_{y\in F}\{d(x, y)\}$

Proof: Suppose that $\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T(x_n) = t$ for some $t \in X$ (10)

If (a) holds, $S(t) = T(t) = t$. Then, the continuity of S and T on F imply $S\big(T(x_n)\big) \to S(t) = t$ and $T\big(S(x_n)\big) \to T(t) = t$, so that, $d\left(S\big(T(x_n)\big), T\big(S(x_n)\big)\right) \to 0$ as desired.

If (b) holds, the compatibility of S and T follow easily from (a) and (10) upon noting that F is closed since S and T are continuous.

Corollary 3.3.4: Suppose that S and T are continuous self maps of $\mathbb{R}$ such that $S - T$ is strictly increasing. If S and T have a common fixed point, then S and T are compatible.

Proof: Immediate, as $S(x_n), T(x_n) \to t$ implies that $F = \{t\}$.

Lemma 3.3.5: Let, $S, T: (X, d, \rho) \to (X, d, \rho)$ be compatible.

(1) If $S(t) = T(t)$, then $S\big(T(t)\big) = T\big(S(t)\big)$.
(2) Suppose that $\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T(x_n) = t$, for some $t \in X$.
    (a) If S is continuous at t, $\lim_{n\to\infty} T\big(S(x_n)\big) = S(t)$.
    (b) If S and T are continuous at t, then, $S(t) = T(t)$ and $S\big(T(t)\big) = T\big(S(t)\big)$.

Proof: Suppose that $S(t) = T(t)$, and let $x_n = t$ for n in $\mathbb{N}$. Then, $S(x_n), T(x_n) \to S(t)$, so that, $d\left(S\big(T(t)\big), T\big(S(t)\big)\right) = d\left(S\big(T(x_n)\big), T\big(S(x_n)\big)\right) \to 0$ by compatibility. Consequently, $d\left(S\big(T(t)\big), T\big(S(t)\big)\right) = 0$ and $S\big(T(t)\big) = T\big(S(t)\big)$, proving (1).

To prove (2)(a), note that if $T(x_n) \to t$, $S(T(x_n)) \to S(t)$ by the continuity of S. But if $S(x_n) \to t$, also since $d\left(T(S(x_n)), S(t)\right) \leq d\left(T(S(x_n)), S(T(x_n))\right) + d\left(S(T(x_n)), S(t)\right)$ the compatibility of S and T require that, $d\left(T(S(x_n)), S(t)\right) \to 0$; i.e., $T(S(x_n)) \to S(t)$.

We prove (2)(b) by noting that $T(S(x_n)) \to S(t)$ by the continuity of S, whereas, $T(S(x_n)) \to T(t)$ by the continuity of T. Thus, $S(t) = T(t)$ by the uniqueness of limit, and therefore $T(S(x_n)) = S(T(x_n))$ by part (1).

Theorem 3.3.6: Let $S\ and\ T$ be continuous compatible maps of a complete digital metric space (X, d, ρ) into itself. Then $S\ and\ T$ have a unique common fixed point in X if there exists an $\alpha \in (0,1)$ and they satisfy

$$S(X) \subset T(X)\ and\ \left(S(x), S(y)\right) \leq \alpha d\left(T(x), T(y)\right)\ for\ all\ x, y \in X \tag{11}$$

Proof: Suppose there is a map S of X into itself which is compatible with T and for which (11) holds. We show that the condition is sufficient to ensure that T and S have a unique fixed point.

To this end, let $x_0 \in X$ and let x_1 be such that $T(x_1) = S(x_0)$. In general, choose x_n so that

$$T(x_n) = S(x_{n-1}) \tag{12}$$

We can do this since $S(X) \subset T(X)$. The relation (11) and (12) imply that $d\left(T(x_{n+1}), T(x_n)\right) \leq \alpha d\left(T(x_n), T(x_{n-1})\right)$ for all n. The lemma yields $t \in X$ such that

$$T(x_n) \to t \tag{13}$$

But then (12) implies that

$$S(x_n) \to t \tag{14}$$

Now, from the definition of compatibility,

$$\lim_{n\to\infty} d\left(S(T(x_n)), T(S(x_n))\right) = 0$$

Now, by continuity of $S\ and\ T$ and the lemma, $S(t) = T(t)$ and $S(T(t)) = T(S(t))$. We can therefore infer

$$d\left(S(t), S(S(t))\right) \leq \alpha d\left(T(t), T(S(t))\right) = \alpha d\left(S(t), S(T(t))\right) = \alpha d\left(S(t), S(S(t))\right).$$

Hence, $d\left(S(t), S(S(t))\right)(1-\alpha) \leq 0$. Since $\alpha \in (0,1)$, $S(t) = S(S(t)) = T(S(t))$, i.e. $S(t)$ is common fixed point of S and T.

To see that S and T can have only one common fixed point, suppose that $x = T(x) = S(x)$ and $y = T(y) = S(y)$. Then (11) implies that $d(x, y) = d\left(S(x), S(y)\right) \leq \alpha d\left(T(x), T(y)\right) = \alpha d(x, y)$, or $d(x, y)(1-\alpha) \leq 0$. Since, $\alpha < 1$, so we infer that $x = y$.

Corollary 3.3.7: Let T and S be continuous compatible maps of a complete digital metric space (X, d, ρ) into itself. Suppose, $S(X) \subset T(X)$. If there exists $\alpha \in (0,1)$ and a positive integer k such that (i) $d\left(S^k(x), S^k(y)\right) \leq \alpha d\left(T(x), T(y)\right)$ for all x and y in X, then T and S have a common fixed point.

Proof: Clearly, S^k is compatible with T and $S^k(X) \subset S(X) \subset T(X)$. Thus, the theorem pertains to S^k and T, so there exists a unique $a \in X$ such that $a = T(a) = S^k(a)$. But then, since T and S are compatible, we can write $S(a) = T\left(S(a)\right) = S^k\left(S(a)\right)$, which says that $S(a)$ is a c.f.p. of T and S^k. The uniqueness of a implies $a = S(a) = T(a)$.

Example 3.3.8: Let us consider the digital metric space (X, d, ρ), with the digital metric defined by $d(x, y) = |x - y|$. Consider the self mapping $T, S: X \to X$ given by, $S(x) = 2 - x^2$ and $T(x) = x^2$. Now, $|S(x_n) - T(x_n)| = 2|1 - x^2| \to 0$ iff $x_n = \pm 1$ and $\lim_n S(x_n) = \lim_n T(x_n) = 1$. So, these are compatible but clearly not weakly commutative, e.g. $x = 0$. Now, $d\left(S(x), S(y)\right) \leq \frac{1}{2} d\left(T(x), T(y)\right)$. Hence, S and T satisfy all the hypotheses of the theorem 3.3.6 and hence have a unique common fixed point. The fixed point of these tow mappings is $x = 1$.

3.4. Weakly Compatible Maps:

Definition 3.4.1: Two maps S and T are said to be weakly compatible if they commute at coincidence points.

Remark 3.4.2: Commutative $\Rightarrow$ weakly commutative $\Rightarrow$ compatible $\Rightarrow$ weakly compatible. But the converse of the above implications does not hold.

Theorem 3.4.3: Let S and T be weakly compatible maps of a complete digital metric space (X, d, ρ) into itself. Then S and T have a unique common fixed point in X if there exists an $\alpha \in (0,1)$ and they satisfy the following two conditions:

$S(X) \subset T(X)$ and $d\left(S(x), S(y)\right) \leq \alpha d\left(T(x), T(y)\right)$ for all $x, y \in X$ (15)

Any one of the subspace $S(X)$ or $T(X)$ is complete (16)

Proof: Suppose, there is a map S of X into itself which is weakly compatible with T and for which (15) holds. We show that the condition is sufficient to ensure that T and S have a unique fixed point.

To this end, let $x_0 \in X$ and let x_1 be such that $T(x_1) = S(x_0)$. In general, choose x_n so that

$$T(x_n) = S(x_{n-1}) \tag{17}$$

We can do this since $S(X) \subset T(X)$. The relation (15) and (16) imply that $d\left(T(x_{n+1}), T(x_n)\right) \leq \alpha d\left(T(x_n), T(x_{n-1})\right)$ for all n. The lemma yields $t \in X$ such that

$$T(x_n) \to t \tag{18}$$

But then (17) implies that

$$S(x_n) \to t \tag{19}$$

Since, either $S(X)$ or $T(X)$ is complete, for certainty assume that $T(X)$ is complete subspace of X then the subsequence of $\{x_n\}$ must converge to a limit in $T(X)$. Call it be t. Let, $u \in T^{-1}(t)$. Then $T(u) = t$ as $\{x_n\}$ is a Cauchy sequence containing a convergent subsequence, therefore the sequence $\{x_n\}$ also convergent implying thereby the convergence of subsequence of the convergent sequence. Now, we show that $S(u) = t$.

On setting $x = x_n, y = u$ in (15), we have

$$d\big(S(x_n), S(u)\big) \le \alpha d\big(T(x_n), T(u)\big)$$

Proceeding limit as $n \to \infty$, we get $d\big(t, S(u)\big) \le \alpha d\big(t, T(u)\big) \Rightarrow d\big(t, S(u)\big) \le 0 \Rightarrow S(u) = t$. Thus, u is a coincidence point of S and T. Since S and T are weakly compatible, it follows that $S\big(T(u)\big) = T\big(S(u)\big)$.

We can therefore infer

$$d\Big(S(u), S\big(S(u)\big)\Big) \le \alpha d\Big(T(u), T\big(S(u)\big)\Big) = \alpha d\Big(S(u), S\big(T(u)\big)\Big) = \alpha d\Big(S(u), S\big(S(u)\big)\Big).$$

Hence, $d\Big(S(u), S\big(S(u)\big)\Big)(1-\alpha) \le 0$. Since $\alpha \in (0,1)$, $S(u) = S\big(S(u)\big) = T\big(S(u)\big)$, i.e. $S(u)$ is common fixed point of S and T.

To see that S and T can have only one common fixed point, suppose that $x = T(x) = S(x)$ and $y = T(y) = S(y)$. Then (11) implies that $d(x,y) = d\big(S(x), S(y)\big) \le \alpha d\big(T(x), T(y)\big) = \alpha d(x,y)$, or $d(x,y)(1-\alpha) \le 0$. Since, $\alpha < 1, x = y$.

Corollary 3.4.4: Let T and S be weakly compatible maps of a complete non Newtonian metric space (X, d, ρ) into itself. Suppose, $S(X) \subset T(X)$. If there exists $\alpha \in (0,1)$ and a positive integer k such that (i) $d\Big(S^k(x), S^k(y)\Big) \le \alpha d\big(T(x), T(y)\big)$ for all x and y in X, (ii) either $S^k(X)\ or\ T(X)$ is complete subspace in X, then T and S have a common fixed point.

Proof: Clearly, S^k is weakly compatible with T and $S^k(X) \subset S(X) \subset T(X)$. Thus, the theorem pertains to S^k and T, so there exists a unique $a \in X$ such that $a = T(a) = S^k(a)$. But then, since T and S are compatible, we can write $S(a) = T\big(S(a)\big) = S^k\big(S(a)\big)$, which says that $S(a)$ is a c.f.p. of T and S^k. The uniqueness of a implies $a = S(a) = T(a)$.

Applications of Common Fixed Point Theorems in Digital Metric Space

The basic idea behind defining digital contractions is to solve the problem of storing the images using less storage memory. Now, common fixed point theorem can come as handy tools when we want to redefine the storage of images. That is, if a memory slot is already in use to store a particular image under a given contraction, and this image is to be replaced by another image than by imposing the commutativity or compatibility(depending on the situation) between the maps we can sort the purpose.

Conclusion

The aim of this paper is to introduce common fixed point theorems for the digital metric spaces, using the commutative or compatible properties of the maps. This concept may come handy in the image processing and redefinement of the image storage. The redefinition of the same slot of memory is a proposed application of the proposed concept of the paper.

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Refrences:

[1] Aamri M, Moutawakil DEl, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 2002, 270, 181-188.

[2] Banach S, Sur les operations dans les ensembles abstraits et leurs applications aux equations integrales, Fund. Math. 1922, 3, 133–181.

[3] Boxer L, Digitally Continuous Functions, Pattern Recognition Letters, 15 (1994), 833-839.

[4] Boxer L, Properties of Digital Homotopy, J. Math. Imaging Vis., 22(2005), 19-26.

[5] Boxer L, A Classical Constructions for The Digital Fundamental Group, J. Math. Imaging Vis., 10(1999), 51-62.

[6] Boxer L, Digital Products, Wedges and Covering Spaces, J. Math. Imaging Vis., 25(2006), 159-171.

[7] Boxer L, Continuous Maps on Digital Simple Closed Curves, Appl. Math., 1(2010), 377-386.

[8] Choudhury BS, Das KP, A new contraction principle in Menger spaces, Acta Mathematica Sinicia, English Series 2008, 24(8), 1379-1386.

[9] Choudhury BS, Kumar S, Asha, Das K, Some fixed point theorems in G-metric spaces, Math. Sci. Lett. 2012, 1(1), 25-31.

[10] Das KM, Naik KV, Common fixed-point theorems for commuting maps on a metric space, Proc. Amer. Math. Soc. 1979, 77, 369-373.

[11] Ege O, Karaca I, Fundamental Properties of Simplicial Homology Groups for Digital Images, American Journal of Computer Technology and Application, 1(2013), 25-42.

[12] Ege O, Karaca I, Lefschetz Fixed Point Theorem for Digital Images, Fixed Point Theory Appl., 2013(2013), 13 pages.

[13] Ege O, Karaca I, Applications of The Lefschetz Number to Digital Images, Bull. Belg. Math. Soc. Simon Stevin, 21(2014), 823-839.

[14] Ege O, Karaca I, Banach Fixed Point Theorem for Digital Images, J. Nonlinear Sci. Appl., 8(2015), 237-245.

[15] Herman GT, Oriented Surfaces in Digital Spaces, CVGIP: Graphical Models and Image Processing, 55(1993), 381-396.

[16] Jungck G, Commuting mappings and fixed point, Amer. Math. Monthly 1976, 83, 261-263.

[17] Jungck G, Compatible mappings and common fixed points, International J. Math. Math. Sci. 1986, 9(4), 771-779.

[18] Jungck G, Common fixed points for non-continuous non-self maps on non-metric spaces, Far East J. Mathe. Sci. 1996, 4 (2), 199-212.

[19] Jungck G, Murthy PP, Cho, YJ, Compatible mappings of type (A) and common fixed points, Math. Japon. 1993, 38, 381 – 390.

[20] Kannan R, Some results on fixed point, Bull. Cal. Mathe. Soc. 1968, 60, 71-76.

[21] Kannan R, Some results on fixed point II, Amer. Mathe. Monthly 1969, 76, 405-408.

[22] Karaca I, Ege O, Some Results on Simplicial Homology Groups of 2D Digital Images, Int. J. Inform. Computer Sci., 1(2012), 198-203.

[23] Kong TY, A Digital Fundamental Group, Computers and Graphics, 13(1989), 159-166.

[24] Rani A, Kumar S, Common fixed point theorems in generalized menger spaces with t-norm t of hadžić-type, Adv. Fixed Point Theory 2013, 3(1), 151-163.

[25] Rani A, Kumar S, Kumar N, Garg SK, Common Fixed Point Theorems for Compatible and Weakly Compatible Maps in G-Metic Spaces, Applied Math. 2012, 3, 1128-1134.

[26] Rosenfeld A, Digital Topology, Amer. Math. Monthly, 86(1979), 76-87.

[27] Sessa S, On a weak commutativity condition of mappings in fixed point considerations, Publications DeL"Institut Mathematique", Nouvelle Serie Tome 1982, 32, 149-153.

[28] Z-Dos`enovic` T, A common fixed point theorem for compatible mappings in fuzzy metric spaces using implicit relation, Acta Math. Hang. 2009, 125(4), 357-368.